ISSN : 2320-2971 (Online)

International e-Journal for Education and Mathematics

www.iejem.org vol. 02, No. 02, (Apr. 2013), pp 25-34

An Existence Results Global Attractivity For Nonlinear Functional Integro-Differential Equations

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Article Info.

Received on 22 Jan. 2013
Revised on 02 Feb. 2013
Accepted on 12 Feb. 2013

Keywords:
Integro-Differential equations nonlinear differential equations, global attractivity, real-valued functions Lipschitz conditions.

A B S T R A C T

In this paper we proved existence results as global attractivity for nonlinear functional Integro-Differential equations under Lipschitz conditions.

1. Introduction

Our existence results include several existence results obtained earlier by I.K.Purnaras(1) Banas and Dhage(2),Dhage(3),Banas and Rzepka(4),Hu and Yan(5) under some weaker Lipschitz conditions. A fixed point theorem of Dhage(6) is used in proving main results.

1. Statement of Problem

Let $\mathbb{R}$ denote a real line and $\mathbb{R}_+$ the set of non negative real numbers and let $I = [0, T]$ for $T > 0$. Consider a nonlinear functional integral equation (in short FDE)

$$\frac{d}{dt}\left(\frac{x(t)}{f\big(t, x(t), x(\alpha(t))\big)}\right) = G(t,t)g\big(t, x(t), x(\eta(t))\big), \quad t \in I \qquad (2.1)$$

where $f: I \times \mathbb{R}^2 \to \mathbb{R} - \{0\}$, $g: I \times \mathbb{R}^2 \to \mathbb{R}$, $G(t,s)$ is a continuous on $I \times I$, $\alpha, \eta \in C(I, \mathbb{R})$ with $\alpha(t), \eta(t) \in [0, t]$ for all , $t \in I$.

By a solution of the FDE (2.1) we means a function $x \in (I, \mathbb{R})$ that satisfies the equation (2.1) where $C(I, \mathbb{R})$ is the space of continuous real-valued functions defined on I.

The FDE (2.1), it is quite general and includes some nonlinear differential equation studied earlier by various authors as special cases.

In this paper, we prove existence results as global attractivity results of the solutions for the FDE (2.1) via a measure theoretic fixed point theorem of Dhage (6). Our problem is placed in the space of continuous real-valued functions defined on I. This way our results improve and generalize the attractivity results of Banas and Rzepka (4), Banas and Dhage (2), Dhage (3) and Hu and Yan (5) under some weaker Lipschitz conditions.

2. Auxiliary Results

Let E be a Banach space and let $\mathcal{P}_p(E)$ denote the class of all nonempty subsets of E with property p. Here p may be p=closed (in short cl), p=bounded (in short bd), P= relatively compact (in short rcp), etc. Thus $\mathcal{P}_{cl}(E), \mathcal{P}_{bd}(E), \mathcal{P}_{c,bd}(E)$ and $\mathcal{P}_{rcp}(E)$ denote respectively the classes of closed, bounded, closed and bounded and relatively compact subsets of E. A function $d_H: \mathcal{P}_p(E) \times \mathcal{P}_p(E) \to$ I defined by

$$d_H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A),\} \qquad (3.1)$$

Satisfies all the conditions of a metric on $\mathcal{P}_p(E)$ and is called a Hausdorff-pompeiu metric on E where $D(a,B) = \inf\{\|a-b\|: b \in B\}$ It is known that the hyper space $(\mathcal{P}_{cl}(E), d_H)$ is a complete metric space.

We need the following definitions:

Definition (3.2): A sequence $\{A_n\}$ of non empty sets in $\mathcal{P}_p(E)$ is said to converges to a set A, called the limiting set, if $d_H(A_n, A) \to 0$ as $n \to \infty$. A mapping $\mu: \mathcal{P}_p(E) \to \mathbb{R}_+$ is called continuous if
for any sequence $\{A_n\}$ in $\mathcal{P}_p(E)$,

$$d_H(A_n, A) \to 0 \Rightarrow |\mu(A_n) - \mu(A)| \to 0 \text{ as } n \to \infty.$$

Definition (3.3): A mapping $\mu: \mathcal{P}_p(E) \to \mathbb{R}_+$ is called nondecreasing if $A, B \in \mathcal{P}_p(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \le \mu(B)$, where $\le$ is a order relation by inclusion in $\mathcal{P}_p(E)$.

Definition (3.4)(Dhage[6]): A function $\mu: \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is called a measure of non compactness if it satisfies

i. $\emptyset \neq \mu^{-1}(0) \subset \mathcal{P}_{rcp}(E)$.
ii. $\mu(\overline{A}) = \mu(A)$, where $\overline{A}$ denotes the closure of A.
iii. $\mu(\text{conv}A) = \mu(A)$, where convA denotes the convex null of A
iv. μ is nondecreasing and
v. If $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{P}_{bd}(E)$ such that $\lim_{n\to\infty} \mu(A_n) = 0$, then the limiting set $A_\infty = \overline{\lim_{n\to\infty} A_n}$ is non-empty.

The family kerμ described in (i) is said to be the kernel of the measure of non compactness μ and $\ker\mu = \{A \in \mathcal{P}_{bd}(E) / \mu(A) = 0\} \subset \mathcal{P}_{rcp}(E)$.

Observe that the limiting set A_∞ from (v) is member of the family kerμ. In fact, since $\mu(A_\infty) \le \mu(\overline{A_n}) = \mu(A)$ for any n, we infer that $\mu(A_\infty) = 0$. This yields that $A_\infty \in$ kerμ. This simple observation will be essential in our further investigations.

We give a useful definition.

Definition (3.5): A mapping $Q: E \to E$ is called D-set Lipschitz if there exists a continuous nondecreasing function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \le Q(\mu(A))$ for all $A \in \mathcal{P}_{bd}(E)$ with $Q(A) \in \mathcal{P}_{bd}(E)$, where $\phi(0) = 0$. Sometimes we call the function ϕ a D-function of ϕ on E. In the special case, when $\phi(r) = kr, k > 0$, Q is called a K–set Lipschitz mapping and if $K < 1$, then Q is called a K–set contraction on E. Further, if $\phi(r) < r$ for $r > 0$, then Q is called a nonlinear D-set contraction on E.

Now we state a key fixed point theorem of Dhage(6) which will be used in follows.

Theorem (3.6)(Dhage 6): Let $\mathcal{C}$ be a non-empty, closed, convex and bounded subset of a Banach space E and let $Q: \mathcal{C} \to \mathcal{C}$ be a continuous and nonlinear D-set contraction. Then Q has a fixed point.

Remark (3.5): Let us denote by Fix(Q) the set of all fixed point of the operator Q which belong to $\mathcal{C}$. It can be shown that the set Fix(Q) existing in theorem (3.6) belongs to the family kerμ. In fact if $(Q) \notin$ kerμ, then $\mu(\mathrm{Fix}(Q)) > 0$ and $Q(\mathrm{Fix}(Q)) = \mathrm{Fix}(Q)$. Now from nonlinear D-set contraction it follows that $\mu\left(Q(\mathrm{Fix}(Q))\right) \le \phi\left(\mu(\mathrm{Fix}(Q))\right)$, which is a contradiction, since $\phi(r) < r$ for $r > 0$. Hence $\mathrm{Fix}(Q) \in$ kerμ.

Our further considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$ and with the standard supremum norm $\|x\| = \sup_{t \in \mathbb{R}_+} |x(t)|$.

For, our purposes we will use the Hausdorff or ball measure of non-compactness in $BC(\mathbb{R}_+, \mathbb{R})$. A handy formula for the Hausdorff measure of non compactness useful in applications is defined as follows. Let us fix a nonempty and bounded subset X of the space
$BC(\mathbb{R}_+, \mathbb{R})$ and $T > 0$. For $x \in X, \epsilon \ge 0$, denote
By $\omega^T(X, \epsilon)$ the modulus of continuity of the function X on the closed and bounded interval $[0, T]$ defined by

$$\omega^T(X, \epsilon) = \sup\{|x(t) - \lambda(s)| : t, s \in [0, T], |t - s| \le \epsilon\}$$

Next, let us put

$$\omega^T(X, \epsilon) = \sup\{\omega^T(X, \epsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon)$$

Finally, we define

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(x)$$

Now for a fixed numbert $\in \mathbb{R}_+$, let us denote

$$X(t) = \{x(t): x \in X\} \text{ and } \|X(t)\| = \sup\{|x(t)|: x \in X\}$$

Finally let us consider the functions μ_p defined on the family $\mathcal{P}_{bd}(X)$ by the formulas

$$\mu_a(X) = \max\{\omega_0(X), \limsup_{t\to\infty} \operatorname{diam} X(t)\} \quad \text{and}$$

$$\mu_b(X) = \max\{\omega_0(X), \limsup_{t\to\infty} \|QX(t)\|\}$$

It can be shown as in [7] that the function μ_a and μ_b are measures of non-compactness in the spaceBC$(\mathbb{R}_+, \mathbb{R})$. The kernels $\ker\mu_a$ and $\ker\mu_b$ of the measures μ_a and μ_b consist of non- empty and bounded subset X of $BC(\mathbb{R}_+, \mathbb{R})$ such that function from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundles formed by functions from X tends to zero at infinity.

The above expressed properly of $\ker\mu_a$ and $\ker\mu_b$ permits us to characterize solutions of the integral equations considered in this paper.

We introduce concepts used in this paper, let Ω be a subset of $BC(\mathbb{R}_+, \mathbb{R})$. Let $Q: BC(\mathbb{R}_+, \mathbb{R}) \to BC(\mathbb{R}_+, \mathbb{R})$ be an operator and consider the following operator equation in E

$$Qx(t) = x(t), \quad \text{for all } t \in \mathbb{R}_+, \tag{3.8}$$

We give different characterizations of the solutions for the operator equation (3.8) on $\mathbb{R}_+$.

Definition (3.9): We say that solutions of equation (3.8) are locally attractive if there exists a closed ball $\overline{\mathcal{B}}_r(x_o)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ for some $x_o \in BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary solution $X = X(t)$ and $Y = Y(t)$ of equation (3.8) belonging to $\overline{\mathcal{B}}_r(x_o) \cap \Omega$ we have that

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0 \tag{3.10}$$

In case when for each $\epsilon > 0$, $\exists\, T > 0$ such that

$$|x(t) - y(t)| \le \epsilon \tag{3.11}$$

For all $x, y \in \overline{\mathcal{B}}_r(x_o) \cap \Omega$ being solutions of (3.8) and for $t \ge T$, we will say that solutions of equation (3.8) are uniformly locally attractive on $\mathbb{R}_+$

Definition (3.12): A solution $x = x(t)$ of equation (3.8) is said to be globally attractive if equation (3.10) holds for each solution $y = y(t)$ of equation (3.8) in Ω.

In the case when the condition (3.10) is satisfied uniformly with respect to the set Ω, we say that solutions of equations (3.8) are uniformly globally attractive on $\mathbb{R}_+$.

3. Main Global Attractivity Results

We prove main local attractivity results of this paper.

We consider the following hypothesis in sequel.

(A_o) The functions $\alpha, \beta, \gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\alpha(t) \geq t$ and $\beta(t) \geq t$ for all $t \in \mathbb{R}_+$.

(A_1) The function $F\colon \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists functions $\ell_1, \ell_2 \in BC(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$|F(t, x_1, x_2) - F(t, y_1, y_2)| \leq \ell_1(t)|x_1 - y_1| + \ell_2(t)|x_2 - y_2|$$

for all $(t, x_1, x_2), (t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Moreover, we assume that $L_1 = \sup_{t\geq 0} \ell_1(t)$.

(A_2) The function $t \mapsto F(t, 0,0)$ is bounded with $F_o = \sup_{t\geq 0}|F(t, 0,0)|$.

(A_3) The function $f\colon \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $k_1 \in BC(\mathbb{R}_+, \mathbb{R}_+)$ and $M \in \mathbb{R} > 0$ such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \frac{k_1(t)\max\{|x_1 - y_1|, |x_2 - y_2|\}}{M + \max\{|x_1 - y_1|, |x_2 - y_2|\}}$$

for all $(t, x_1, x_2), (t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Moreover, we assume that $K_1 = \sup_{t\geq 0} k_1(t)$.

(A_4) The function $t \mapsto f(t, 0,0)$ is bounded with $f_o = \sup_{t\geq 0}|f(t, 0,0)|$.

(A_5) There exists a continuous function $b\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $|g(t, s, x)| \leq b(t, s)$ for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover, we assume that $\lim\limits_{t\to\infty} \int_0^{\beta(t)} b(t, s)ds = 0$.

Remark (4.1): The Lipschitz condition given for the nonlinearity f in the hypothesis (A_3) is more general than the usual Lipschitz condition. In fact, if $k_1 < M$ then it reduces to the Lipschitz condition of the function f

$$|f(t, x) - f(t, y)| \leq k_1(t)|x - y|.$$

Remark (4.2): Note that the function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\omega(t) = \int_0^{\beta(t)} b(t, s)ds$ is continuous.

This together with the hypothesis (A_5) implies that the number $W = \sup_{t\geq 0} \omega(t)$ is finite.

Theorem(4.3): Assume that the hypothesis $(A_0) - (A_5)$ holds. Further if $L_1 \leq M$, then the FDE(2.1) has solution and which are uniformly globally attractive on $\mathbb{R}_+$.

Proof: Consider a nonlinear functional differential equation FDE (2.1)

$$\frac{d}{dt}\left(\frac{x(t)}{f\big(t, x(t), x(\alpha(t))\big)}\right) = G(t, t)g\Big(t, x(t), x\big(\eta(t)\big)\Big), \quad t \in I$$

Converting FDE (2.1) into functional integral equation (In short FIE)

$$x(t) = f\Big(t, x(t), x\big(\alpha(t)\big)\Big) \int_0^{\beta(t)} G(t, s)g\Big(t, x(s), x\big(\eta(s)\big)\Big)\, ds \qquad (4.4)$$

Set $E = BC(\mathbb{R}_+, \mathbb{R}_+)$. Define a mapping $Q\colon E \to E$ by

$$Qx(t) = F\left(t, f\left(t, x(t), x\left(\alpha(t)\right)\right), \int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds\right) \text{ for } t \in I = \mathbb{R}_+ \quad (4.5)$$

First we show that Q maps E into itself. As all the functions on the right hand side of (4.5) are continuous, the function Qx is continuous on $\mathbb{R}_+$ for each $x \in E$. Again, by hypothesis $(A_1) - (A_2)$, we obtain

$$|Qx(t)| \le \left|F\left(t, f\left(t, x(t), x\left(\alpha(t)\right)\right), \int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds\right)\right|$$

$$\le \left|F\left(t, f\left(t, x(t), x\left(\alpha(t)\right)\right), \int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds\right) - F(t,o,o)\right| + |F(t,o,o)|$$

$$\le \ell_1(t)\left[\left|f\left(t, x(t), x\left(\alpha(t)\right)\right) - f(t,0,0)\right| + |f(t,0,0)|\right]$$

$$+ \ell_2(t)\left|\int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds\right| + F_0$$

$$\le \frac{\ell_1(t)k_1(t)\max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}}{M + \max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}} + \ell_1(t)f_0 + \ell_2(t)\left|\int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds\right| + F_0$$

$$\le \frac{L_1 K_1 \|x\|}{M + \|x\|} + L_1 f_0 + \ell_2(t)\int_0^{\beta(t)} b(t,s) ds + F_0$$

$$\le L_1K_1 + L_1 f_0 + \omega(t) + F_0$$

$$\le L_1K_1 + L_1 f_0 + W + F_0 = r \quad \text{for all } t \in \mathbb{R}_+.$$

This shows that Qx is bounded function on $\mathbb{R}_+$. Next, we show that Q satisfies all the conditions of theorem (3.6).

Define a closed ball $\overline{\mathcal{B}}_r(0)$ in E centered at the origin of radius $r = L_1K_1 + L_1f_0 + W + F_0$. Then Q defines a map $Q: E \to \overline{\mathcal{B}}_r(0)$ and, in particular $Q: \overline{\mathcal{B}}_r(0) \to \overline{\mathcal{B}}_r(0)$. Because of this fact solution of the FIE (3.4) automatically FDE (2.1), if they exist, are global in nature.

Next, we show that Q is continuous on $\overline{\mathcal{B}}_r(0)$. Let $\epsilon > 0$, since (A_3) holds, there is a real number $T > 0$ such that $\omega(t) = \frac{\epsilon}{2L_2}$ for all $t \ge T$. Now we consider the following two cases.

Case(I): Let $t \in \mathbb{R}_+$ be such that $t \ge T$ and let $x, y \in \overline{\mathcal{B}}_r(0)$ be such that $\|x - y\| \le \epsilon$. Then ,we have

$$|Qx(t) - Qy(t)| \le \left|F\left(t, f\left(t, x(t), x\left(\alpha(t)\right)\right), \int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds\right) - F\left(t, f\left(t, y(t), y\left(\alpha(t)\right)\right), \int_0^{\beta(t)} G(t,s) g\left(t, y(s), y\left(\eta(s)\right)\right) ds\right)\right|$$

$$\le \ell_1(t)k_1(t)\left|f\left(t, x(t), x\left(\alpha(t)\right)\right) - f\left(t, y(t), y\left(\alpha(t)\right)\right)\right|$$

$$+ \left|\int_0^{\beta(t)} G(t,s) g\left(t, x(s), x\left(\eta(s)\right)\right) ds - \int_0^{\beta(t)} G(t,s) g\left(t, y(s), y\left(\eta(s)\right)\right) ds\right|$$

$$\le \frac{\ell_1(t)k_1(t)\max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}}{M + \max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}} + \ell_2(t)\int_0^{\beta(t)} |G(t,s)|\left|g\left(t, x(s), x\left(\eta(s)\right)\right)\right| ds$$

$$+ \ell_2(t)\int_0^{\beta(t)} |G(t,s)|\left|g\left(t, y(s), y\left(\eta(s)\right)\right)\right| ds$$

$$\leq \frac{L_1K_1\|x-y\|}{M+\|x-y\|} + 2\ell_2(t)\int_0^{\beta(t)} b(t,s)ds$$

$$\leq \frac{L_1K_1\|x-y\|}{M+\|x-y\|} + 2\ell_2\omega(t) = 2\epsilon \qquad (4.6)$$

Case (II): Now for any $t \in [0,T]$ and $x, y \in [-r,r]$ with $|x-y| \leq \epsilon$, one has

$|Qx(t) - Qy(t)|$

$$\leq \frac{L_1K_1\|x-y\|}{M+\|x-y\|} + L_2\int_0^{\beta(t)} |G(t,s)|\left|g\Big(t,x(s),x\big(\eta(s)\big)\Big) - g\Big(t,y(s),y\big(\eta(s)\big)\Big)\right| ds$$

$$\leq 2\epsilon + L_2\int_0^{\beta_T}\omega(g,\epsilon)ds \qquad (4.7)$$

Where $\beta_T = \sup\{\beta(t): t \in [0,T]\}$ and

$$W(g,\epsilon) = \sup\left\{\left|g\Big(t,x(s),x\big(\eta(s)\big)\Big) - g\Big(t,y(s),y\big(\eta(s)\big)\Big)\right| : t,s \in [0,T], x,y \in [-r,r], |x-y| \leq \epsilon\right\}$$

Thus

$$|Qx(t) - Qy(t)| \leq \max\left\{2\epsilon, 2\epsilon + L_2\int_0^{\beta_T}\omega(g,\epsilon)ds\right\} \qquad (4.8)$$

Now, $W(g,\epsilon) \to 0$ as $\epsilon \to 0$ and hence from (4.8) it follows that Q is a continuous on $\bar{\mathcal{B}}_r(0)$ into itself.

Next, we show that Q is k-set continuous on $\bar{\mathcal{B}}_r(0)$.

Let $\epsilon > 0$ and $T > 0$ be a fixed real number. Choose a function $x \in \bar{\mathcal{B}}_r(0)$ and $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \leq \epsilon$. Then in view of our assumptions $(A_1) - (A_3)$,

$$|Qx(t_1) - Qx(t_2)| \leq$$

$$\left|F\Big(t_1, f\big(t_1, x(t_1), x(\alpha(t_1))\big), \int_0^{\beta(t_1)} G(t_1,s)g\big(t_1,x(s),x(\eta(s))\big)ds\Big) - F\Big(t_2, f\big(t_2, x(t_2), x(\alpha(t_2))\big), \int_0^{\beta(t_2)} G(t_2,s)g\big(t_2,x(s),x(\eta(s))\big)ds\Big)\right|$$

$$\begin{aligned}
\leq &\left|F\Big(t_1, f\big(t_1, x(t_1), x(\alpha(t_1))\big), \int_0^{\beta(t_1)} G(t_1,s)g\big(t_1,x(s),x(\eta(s))\big)ds\Big)\right.\\
&\left.- F\Big(t_1, f\big(t_2, x(t_2), x(\alpha(t_2))\big), \int_0^{\beta(t_2)} G(t_2,s)g\big(t_2,x(s),x(\eta(s))\big)ds\Big)\right|\\
&+\left|F\Big(t_1, f\big(t_2, x(t_2), x(\alpha(t_2))\big), \int_0^{\beta(t_2)} G(t_2,s)g\big(t_2,x(s),x(\eta(s))\big)ds\Big)\right.\\
&\left.- F\Big(t_2, f\big(t_2, x(t_2), x(\alpha(t_2))\big), \int_0^{\beta(t_2)} G(t_2,s)g\big(t_2,x(s),x(\eta(s))\big)ds\Big)\right|
\end{aligned}$$

$$\begin{aligned}
\leq\ &\ell_1(t)\left|f\big(t_1, x(t_1), x(\alpha(t_1))\big) - f\big(t_2, x(t_2), x(\alpha(t_2))\big)\right|\\
&+ \ell_2(t)\left|\int_0^{\beta(t_1)} G(t_1,s)g\big(t_1,x(s),x(\eta(s))\big)ds\right.\\
&\left.- \int_0^{\beta(t_2)} G(t_2,s)g\big(t_2,x(s),x(\eta(s))\big)ds\right| + w_r^T(F,\epsilon)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \ell_1(t)\left|f\Big(t_1,x(t_1),x\big(\alpha(t_1)\big)\Big)-f\Big(t_1,x(t_2),x\big(\alpha(t_2)\big)\Big)\right| \\
&+\ \ell_1(t)\left|f\Big(t_1,x(t_2),x\big(\alpha(t_2)\big)\Big)-f\Big(t_2,x(t_2),x\big(\alpha(t_2)\big)\Big)\right| \\
&+\ell_2(t)\left|\int_0^{\beta(t_1)} G(t_1,s)g\Big(t_1,x(s),x\big(\eta(s)\big)\Big)\,ds\right. \\
&\left.-\int_0^{\beta(t_1)} G(t_2,s)g\Big(t_2,x(s),x\big(\eta(s)\big)\Big)\,ds\right| \\
&+\ell_2(t)\left|\int_0^{\beta(t_1)} G(t_2,s)g\Big(t_2,x(s),x\big(\eta(s)\big)\Big)\,ds\right. \\
&\left.-\int_0^{\beta(t_2)} G(t_2,s)g\Big(t_2,x(s),x\big(\eta(s)\big)\Big)\,ds\right| + w_r^T(F,\epsilon) \\
\leq\ & \frac{\ell_1(t)k_1(t)\max\{|x-y|,|x(\alpha(t))-y(\alpha(t))|\}}{M+\max\{|x-y|,|x(\alpha(t))-y(\alpha(t))|\}} + L_1 w_r^T(F,\epsilon) \\
&+\ell_2(t)\left|\int_{\beta(t_2)}^{\beta(t_1)} |G(t_2,s)|\left|g\Big(t_2,x(s),x\big(\eta(s)\big)\Big)\right| ds\right| \\
&+\ell_2(t)\int_0^{\beta_T}\left|G(t_1,s)g\Big(t_1,x(s),x\big(\eta(s)\big)\Big)-G(t_2,s)g\Big(t_2,x(s),x\big(\eta(s)\big)\Big)\right| ds \\
&+w_r^T(F,\epsilon) \\
\leq\ & \frac{L_1K_1w^T\big(X,\omega^T(\alpha,\epsilon)\big)}{M+w^T\big(X,\omega^T(\alpha,\epsilon)\big)} + L_1w_r^T(f,\epsilon) + L_2\int_0^{\beta_T}\omega_r^T(g,\epsilon)ds + L_2G_r^T\epsilon + w_r^T(F,\epsilon)
\end{aligned} \tag{4.9}$$

Where

$$w_r^T(f,\epsilon)=\sup\left\{\left|f\Big(t_1,x(t_1),x\big(\alpha(t_1)\big)\Big)-f\Big(t_2,x(t_2),x\big(\alpha(t_2)\big)\Big)\right| : t_1,t_2\in[0,T],|t_1-t_2| \leq \epsilon, x\in[-r,r]\right\}.$$

$$\omega_r^T(g,\epsilon)=\sup\left\{\left|G(t_1,s)g\Big(t_1,x(s),x\big(\eta(s)\big)\Big)-G(t_2,s)g\Big(t_2,x(s),x\big(\eta(s)\big)\Big)\right| : t_1,t_2\in[0,T], |t_1-t_2|\leq\epsilon, x\in[-r,r]\right\}$$

$$w_r^T(F,\epsilon) = \sup\left\{|F(t_1,x,y)-F(t_2,x,y)| : t_1,t_2\in[0,T], |t_1-t_2|\leq\epsilon, x\in[-(k_1+f_0),(k_1+f_0)], y\in[-w,w]\right\}.$$

And

$$G_r^T=\sup\left\{\left|g\Big(t,x(s),x\big(\eta(s)\big)\Big)\right| : t,s\in[0,T], x\in[-r,r]\right\}.$$

From the estimate (4.9) it follows that

$$W^T(QX,\epsilon)\leq\frac{L_1K_1w^T(X,\epsilon)}{M+w^T(X,\epsilon)}+L_1w_r^T(f,\epsilon)+L_2\int_0^T\omega_r^T(g,\epsilon)ds+L_2G_r^T\epsilon+w_r^T(F,\epsilon) \tag{4.10}$$

Since the functions f, g are continuous on $[0,T]\times[0,T]\times[-r,r]$, They are continuous there, and therefore we have that $W^T(\alpha,\epsilon)\to 0$, $w_r^T(f,\epsilon)\to 0$ and $\omega_r^T(g,\epsilon)\to 0$ as $\epsilon\to 0$. Therefore, from (4.10) it follows that

$W_r^T(Qx)\leq\frac{L_1K_1w_0^T(X)}{M+w_0^T(X)}$ this further implies that

$$W_0(Qx)\leq\frac{L_1K_1w_0(X)}{M+w_0(X)} \tag{4.11}$$

Now, Let X be a non empty subset of $\overline{\mathcal{B}}_r(0)$, then for any $x, y \in X$ and $t \in \mathbb{R}_+$,

$$|Qx(t) - Qy(t)|$$
$$\leq \ell_1(t)\left|f\Big(t, x(t), x\big(\alpha(t)\big)\Big) - f\Big(t, y(t), y\big(\alpha(t)\big)\Big)\right| + \ell_2(t)\int_0^{\beta(t)} |G(t,s)|\left|g\Big(t, x(s), x\big(\eta(s)\big)\Big)\right| ds + \ell_2(t)\int_0^{\beta(t)} |G(t,s)|\left|g\Big(t, y(s), y\big(\eta(s)\big)\Big)\right| ds$$

$$\leq \frac{\ell_1(t)k_1(t)\max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}}{M + \max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}} + \ell_2(t)\int_0^{\beta(t)} |G(t,s)|\left|g\Big(t, x(s), x\big(\eta(s)\big)\Big)\right| ds + \ell_2(t)\int_0^{\beta(t)} |G(t,s)|\left|g\Big(t, y(s), y\big(\eta(s)\big)\Big)\right| ds$$

$$\leq \frac{\ell_1(t)k_1(t)\max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}}{M + \max\{|x-y|, |x(\alpha(t)) - y(\alpha(t))|\}} + 2\ell_2(t) \qquad (4.12)$$

Hence

$$\text{daimQX}(t) \leq \frac{L_1K_1\text{daimX}(t)}{M + \text{daimX}(t)} + 2L_2w(t)$$

Taking the limit superior as $t \to \infty$ in the above inequality yields

$$\limsup_{t\to\infty} \text{daimQX}(t) \leq \frac{L_1K_1 \limsup_{t\to\infty}\{\text{daimX}(t), \text{daimX}(\alpha(t))\}}{M + \limsup_{t\to\infty}\{\text{daimX}(t), \text{daimX}(\alpha(t))\}}$$
$$\leq \frac{L_1K_1 \lim_{t\to\infty} \sup\text{daimX}(t)}{M + \lim_{t\to\infty} \sup\text{daimX}(t)} \qquad (4.13)$$

Further, using the measure of noncompactness μ_0 defined by formula (3.8) and keeping in mind the estimate (4.11) and (4.13), we obtain

$$\mu_a(QX) = \max\{w_0(QX), \lim_{t\to\infty} \sup \text{daimQX}(t)\}$$
$$\leq \max\left\{\frac{L_1K_1w_0^T(X)}{M + w_0^T(X)}, \frac{L_1K_1 \limsup_{t\to\infty} \text{daimX}(t)}{M + \limsup_{t\to\infty} \text{daimX}(t)}\right\}$$
$$\leq \frac{L_1K_1\max\left\{w_0^T(X), \limsup_{t\to\infty} \text{daimX}(t)\right\}}{M + \max\left\{w_0^T(X), \limsup_{t\to\infty} \text{daimX}(t)\right\}}$$
$$= \frac{L_1K_1\mu_a(X)}{M + \mu_a(X)} \qquad (4.14)$$

From the above estimate we infer that $\mu_a(QX) \leq \Psi\big(\mu_a(X)\big)$ where $\Psi(r) = \frac{L_1K_1r}{M+r}$. Hence we apply theorem (3.6) to deduce that Q has a fixed point x in the ball $\overline{\mathcal{B}}_r(0)$. Obviously x is a solution of the FIE (4.4) automatically FDE (2.1). Moreover, taking into account that the image of the space E under the operator Q is contained in the ball $\overline{\mathcal{B}}_r(0)$ we infer that the set Fix(Q) of all fixed points of Q in E is contained in ball $\overline{\mathcal{B}}_r(0)$. Obviously, the set Fix(Q) contains all solution of the FIE (4.4) as well as FDE (2.1). On the other hand from remark (3.7), we conclude that the $\mathrm{Fix}(Q)$ belongs to the family $\ker\mu_a$. Now taking into account the description of sets belonging to $\ker\mu_a$, we deduce that all solution for the FIE (4.4) as well as FDE (2.1) are globally uniformly attractive on $\mathbb{R}_+$. This completes the proof.

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