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Application Of Dimensional Analysis To Solve Ordinary Differential Equation and Partial Differential Equation

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A B S T R A C T

In this paper we will try to explain that how the method "Dimensional Analysis" works in solving Ordinary Differential equation and Partial Differential Equation through different examples. For this purpose we will follow two different tactics, the first of them will consist in making a mathematical hypothesis on the behaviour of this unknown function and we will justify (as far as possible) such assumption. In the second of our tactics we will use the so called dimensional discrimination. Such tactic allows us to obtain a complete solution for our problems and it is not needed to make any previous assumption.

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1. Introduction

Dimensional Analysis has usually been employed in different areas such as problems; fluid mechanics etc… and these problems are always described by partial differential equations (see [1]-[7]). This method helps us to reduce the number of quantities that appear into an equation and to obtain ordinary differential equations. We would like to point out that this tool is more effective if one practices the spatial discrimination, such tactic allows us to obtain better results than with the standard application of Dimensional Analysis. Knowing that Dimensional Analysis works well in partial differential equation we would like to extend this method to the study of Ordinary Differential Equation (the first order in this case) in a systematic way. There are in the literature previous work in this direction for Ordinary Differential Equation of first order.

Our purpose in this work is to explain, through examples, how Dimensional Analysis works in order to find these changes of variables in a trivial way, i.e. without the knowledge of the symmetries of the ode under study. The idea is as follows. When we are studding an Ordinary Differential Equation from the dimensional point of view, we must require that such Ordinary Differential Equation verifies the principle of dimensional homogeneity i.e. that each term within the equation have the same dimensions, for examples, speed, or energy density. To clarify this concept we consider the following ode, $y' = \frac{y}{x} + x$, where each term must have dimensions of y' i.e. $[y'] = yx^{-1}$, where $[.]$ stands for dimensional equation of the quantity. As we can see, this Ordinary Differential Equation does not verify the principle of dimensional homogeneity, since the term x has dimensions of x, i.e. $[x] = x$. In order to do that we shall rewrite the Ordinary Differential Equation with some constant $[a] = yx^{-2}$,so that the given Ordinary Differential Equation now verifies the principle of dimensional homogeneity. We would like to emphasize that this situation does not appear when one is studying physical or engineering problems since (as it supposed) that such problems (equations) verify the principle of dimensional homogeneity and we do not need to introduce new dimensional constants, that must have physical meaning, for example, the coefficients, etc…

Precisely this dimensional constant suggests us the change of variable $(x, y(x)) \rightarrow (t, u(t))$ Where $(t = x, u = yx^{-2})$ in such a way that rewriting the original Ordinary Differential Equation in these new variables we get an variable Separable form of an Ordinary Differential Equation as $u + u't = 1$. The reason is the following. We know from the Lie group theory that if it is known symmetry of an ode then this symmetry brings us through a change of variables to obtain a simpler Ordinary Differential Equation (a quadrature or a Ordinary Differential Equation with separating variables) and therefore the solution is found in a closed form. To find this change of variables it is used the invariants that generate each symmetry, i.e. the first principle is that it is useful to pass to new coordinates such that one of the coordinate functions is an invariant of the group. After such transformation it often (but not always) happens that the variables separate and the equation can be solved in closed form. We must stress that taking the new dependent variables to be variables is also very important.

2.1.1Example: Consider a first order homogenous differential equation $\left(x^2+y^2\right)\,dx = 2xy\,dy$. (1)

Solution.

(A) Traditional method.

Taking the change of variable. u = y/x we have:

$u' = \frac{1 - u^2}{2ux} \Rightarrow \frac{2u\,du}{1 - u^2} = \frac{dx}{x}$ Taking integration both the sides, we get

$$\ln\left(1-u^2\right) = \ln x \Rightarrow y^2 = x^2 + x, \quad (2)$$

(B) Dimensional Analysis.

We go to next consider equation. (1), written as follows

$$y' = \frac{\left(a^2x^2 + y^2\right)}{2xy}\ , \quad (3)$$

Where the dimensional constant a , makes the ode verify the dimensional principle of homogeneity if

$$[a] = \left[\frac{y}{x}\right] = x^{-1}y\ , \quad (4)$$

Where $[.]$ stands for the dimensional equation of the quantity.

Applying the pi theorem we obtain the dimensionless variables that help us to simplify the original ode. Therefore taking into the following dimensional matrix

we take

	y	x	a

x	0	1	-1
y	1	0	1

$$\Rightarrow \quad \pi_1 = \frac{a\mathrm{x}}{\mathrm{y}}\ , \tag{5}$$

$$\Rightarrow \mathrm{y} = a\mathrm{x}$$

It is easy to verify that equation (5) is a particular Solution of equation (1).

Now let us introduce new variables t and u (t) as follow:

$$\left(\mathrm{t} = \mathrm{x},\ \mathrm{u}(\mathrm{t}) = a\ \frac{\mathrm{x}}{\mathrm{y}}\right), \quad \Rightarrow \left(\mathrm{x} = \mathrm{t}, \mathrm{y} = a\,\frac{\mathrm{t}}{\mathrm{u}(\mathrm{t})}\right) \tag{6}$$

Now, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a}{u} + t\,a\ (-1)\,u^{-2}\ u'$

This change of variables brings us to rewrite equation (3) as follows:

$$\Rightarrow \quad \frac{a}{u} - \frac{t\,a}{u^2}\,u' = \frac{a^2\,t^2 + \left(\frac{t^2\,a^2}{u^2}\right)}{2\,t\left(\frac{t\,a}{u}\right)}$$

$$\Rightarrow \quad \frac{u'}{u\left(1 - u^2\right)} = \frac{1}{2t} \quad \Rightarrow \quad \ln\left|\frac{u}{\sqrt{1 - u^2}}\right| = \frac{1}{2}\ln t, \tag{7}$$

And hence

$$y^2 = a^2\,x^2 + x\,c_1. \tag{8}$$

As we can see in this trivial example, the Dimensional analysis induces a change of variable. This enables us to obtain an ordinary differential equation simpler than the original.

We can also think in the following way

$$y' = b\ \frac{x}{2\,y} + a\ \frac{y}{2\,x}, \tag{9}$$

Where $[a] = x,\qquad [b] = y^2\ x^{-1},$ (10)

And hence,

$$\left(t = \frac{x}{a}\ ,\ u(t) = \frac{y}{\sqrt{b\,x}}\right) \quad \Rightarrow \quad \left(x = at\ \ ,\ \ y = u(t)\sqrt{a\ b\ t}\right) \tag{11}$$

Again, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$$= \frac{u'}{a}(\ ab\ )^{\frac{1}{2}}\ t^{\frac{1}{2}} + u\ (ab)^{\frac{1}{2}}\frac{1}{2a}\ t^{-\frac{1}{2}},$$

$$\therefore\ \frac{dy}{dx} = \frac{(\ ab\ )^{\frac{1}{2}}}{2a}\left(\ u'\,t^{\frac{1}{2}} + u\ t^{\frac{1}{2}}\right)$$

Putting $\frac{dy}{dx}$ in equation (1), and simplifying we get.

$$u\ du\ =\ \frac{dt}{a}$$

Integrating, we get

$$\Rightarrow\ \frac{u^2}{2}\ =\ \frac{t}{a} + c$$

Put $u = \frac{y}{\sqrt{b\,x}}$ and $\frac{x}{a} = t$ in above equation, we get

$$\Rightarrow\ \frac{y^2}{2bx}\ =\ \frac{x}{a^2} + c$$

$$\Rightarrow\ a^2y^2\ =\ 2bx^2 + 2bx\ c \tag{12}$$

Once we have obtain the solution, without loss of generality, the constants a, b can be assumed as a = 1 and b = ½ and thus we get,

$$y^2\ =\ x^2 + x\ c$$

2.2 The Method Of Dimensional Analysis; (for Partial Differential Equation)

The application of the method of dimensional analysis for finding similarity transformations has been nicely demonstrated by Sedov [5]. Generalizations to the method have been proposed by Moran [4]. In this section, we will use the dimensional analysis procedure as suggested by Moran [4] Which we shall refer to as the "Generalized Dimensional Analysis".

The success of dimensional method depends on the proper identification of physical parameters and variables that go into the description of a physical problem. As an illustration, we will consider the following boundary value problem commonly known as the Rayleigh Flow problem. An infinite plate is immersed in an incompressible fluid at rest. The plate is suddenly accelerated, so that it moves parallel to itself at a constant velocity, U_0.Let u is the fluid velocity in the x-direction, v and w the velocities in the y and z directions, respectively. From physical symmetry v = w = 0, and the viscous-diffusion equation describing the flow can be written as

$$\upsilon \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{13}$$

Where υ kinematic viscosity and t is the time.

The boundary and initial conditions are $u(\upsilon, o) = 0$

$$u(0, t) = U_0 \qquad t > 0$$

$$u(y, 0) = 0 \qquad y > 0$$

$$u(\infty, t) = 0$$

In the method of dimensional analysis used here, we will distinguish between lengths in different directions by assigning for each direction a separate dimension. By doing so, we will not lose the physical information that would be needed to discover the similarity transformation. The velocity in the z direction, u can be expressed as

$$u = f(y, \upsilon, t, U_0) \tag{14}$$

The "dimensional matrix" can now be written as

	u	y	t	υ	U_0
M	0	0	0	0	0
L_x	1	0	0	0	1
L_y	0	1	0	2	0
T	-1	0	1	-1	-1
	a_1	a_2	a_3	a_4	a_5

The rank of the matrix is 3 and the number of variables is 5. Therefore, the numbers of Pi terms are $5 - 3 = 2$.

The "dimensions" a_i in the dimensional matrix are defined by the relationship

Where the equal sign means "dimensionally equal to"

The dimensional matrix is equivalent to the following system of equation

$$\begin{aligned} a_1 + a_5 &= 0 \\ a_1 + 2a_4 &= 0 \\ -a_1 + a_3 - a_4 - a_5 &= 0 \end{aligned} \qquad (15)$$

Solving equation. (15) And rewriting equation. (14)

$$\left(\frac{u}{U_0}\right)^{a_1}\left[\frac{y}{(\nu t)^{\frac{1}{2}}}\right]^{a_2} = M^0 L_x^{\,0} L_y^{\,0} T^0$$

Where again, the equal sign means 'dimensionally equal to'.

The Pi-terms are therefore $\pi_1 = \dfrac{u}{U_0}$; $\pi_2 = \dfrac{y}{(\upsilon t)^{\frac{1}{2}}}$

The similarity transformation can now be written as

$$\pi_1 = f(\pi_2)$$

or

$$\frac{u}{U_0} = f\left(\frac{y}{(\upsilon t)^{\frac{1}{2}}}\right)$$

Substituting above transformations in equation (13) then it can be easily transformed in to ordinary differential equation.

2.2.1 Laminar Boundary Layer Equation: Dimensional Method

Consider the problem of an incompressible boundary layer flow, on a flat plate with a uniform free stream U_0. The flow is governed by the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{16}$$

And the momentum equation $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}$ (17)

The boundary conditions are:

$$y=0,\ u=v=0,\ y\rightarrow\infty,\ u\rightarrow U_0 \tag{18}$$

Assume the dimensional relationship as $u=u\left(x\ ,y\ ,\upsilon,U_0\right)$, $v=v\left(x\ ,y\ ,\upsilon\ ,U_0\right)$.

The dimensional matrix for the relationship

$$u^{a_1}\ x^{a_2}\ y^{a_3}\ \upsilon^{a_4}\ U_0^{a_5}=M^0\ L_y^0\ L_y^0\ T^0 \tag{19}$$

	u	x	y	υ	U_0
M	0	0	0	0	0
L_x	1	1	0	0	1
L_y	0	0	1	2	0
T	-1	0	0	-1	-1
	a_1	a_2	a_3	a_4	a_5

$$a_1+a_2+a_5=0 \tag{20}$$

$$a_3+2a_4=0 \tag{21}$$

$$-a_1 - a_4 - a_5 = 0 \tag{22}$$

Solving (20) and (22) : $\Rightarrow a_2 - a_4 = 0$
$\Rightarrow a_2 = a_4$

Again from equation (21)

$$\Rightarrow a_3 = -2a_4 = -2a_2$$

$$\Rightarrow a_2 = -\frac{1}{2}a_3$$

$$\Rightarrow a_4 = -\frac{1}{2}a_3$$

$$\Rightarrow a_5 = -a_1 - a_4$$

From equation (22) $\Rightarrow a_5 = -a_1 - a_2$

$$= -a_1 + \frac{1}{2}a_3$$

Putting a_2, a_4, and a_5 in equation in (19)

$$\Rightarrow u^{a_1}\ x^{-\frac{1}{2}a_3}\ y^{a_3}\ \upsilon^{-\frac{1}{2}a_3}\ U_0^{-a_1+\frac{1}{2}a_3} = M^0\ L_y^0\ L_y^0\ T^0$$

$$\Rightarrow \left(\frac{u}{U_0}\right)^{a_1}\left[y\left(\frac{U_0}{x\,\upsilon}\right)^{\frac{1}{2}}\right] = M^0\ L_y^0\ L_y^0\ T^0$$

let $\pi_1 = \left(\frac{u}{U_0}\right)$, $\pi_2 = y\left(\frac{U_0}{x\upsilon}\right)^{\frac{1}{2}}$

The similarity Transformation can be now written as $\pi_1 = f(\pi_2)$

$$\frac{u}{U_0} = f_1\left(y\left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}}\right) \tag{23}$$

Similarly, for $v = v\ (x, y, \upsilon, U_0)$ the dimensional matrix for the relationship

$$v^{a_1}\ x^{a_2}\ y^{a_3}\ \upsilon^{a_4}\ U_0^{a_5} = M^0\ L_y^0\ L_y^0\ T^0 \tag{24}$$

	V	x	y	υ	U_0
M	0	0	0	0	0

L_x	0	1	0	0	1
L_y	1	0	1	2	0
T	-1	0	0	-1	-1
	a_1	a_2	a_3	a_4	a_5

$$a_2 + a_5 = 0$$

$$a_1 + a_3 + 2a_4 = 0$$

$$- a_1 - a_4 - a_5 = 0$$

Solving above three equations we get the following relations

$$a_4 = -\frac{1}{2}a_1 - \frac{1}{2}a_3$$

$$a_2 = \frac{1}{2} a_1 - \frac{1}{2} a_3$$

and

$$a_5 = - \frac{1}{2} a_1 + \frac{1}{2} a_3$$

Putting a_2, a_4, and a_5 in equation (6) we get

$$\nu^{a_1}\, x^{-\frac{1}{2}a_3+\frac{1}{2}a_1}\, y^{a_3}\, \upsilon^{-\frac{1}{2}a_1-\frac{1}{2}a_3}\, U_0^{-\frac{1}{2}a_1+\frac{1}{2}a_3} = M^0\, L_x^{\,0}\, L_y^{\,0}\, T^0$$

$$\Rightarrow \left[v \left(\frac{x}{\upsilon U_0}\right)^{\frac{1}{2}} \right]^{a_1} \left[y \left(\frac{U_0}{x\upsilon}\right)^{\frac{1}{2}} \right]^{a_3} = M^0\, L_x^{\,0}\, L_y^{\,0}\, T^0$$

Let

$$\pi_1 = v \left(\frac{x}{\upsilon U_0}\right)^{\frac{1}{2}} \;,\; \pi_1 = y \left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}}$$

As $\pi_1 = g\,(\pi_2)$

$$\Rightarrow v \left(\frac{x}{\upsilon\, U_0}\right)^{\frac{1}{2}} = g \left[y \left(\frac{U_0}{\upsilon x}\right)\right]^{\frac{1}{2}} \tag{25}$$

Equation (24) and (25) are the similarity transformation.

$$\text{i.e. } u = U_0 f_1\left(y\left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}}\right) \qquad (26)$$

$$\Rightarrow u = U_0 f_1(\phi), \qquad \text{where } \phi = y\left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}}$$

$$\& \quad v = \left(\frac{\upsilon U_0}{x}\right)^{\frac{1}{2}} g_1(\phi) \qquad (27)$$

Substituting transformations (26) and (27) in (16) and (17), we get,

$$\Rightarrow u\left(-\frac{1}{2}\left(\frac{U_0}{x}\right)\phi f_1^{'}\right) + v\left(U_0 f_1^{'}\left(\frac{U_0}{\upsilon x}\right)^{\frac{1}{2}}\right) = \upsilon U_0\left(\frac{U_0}{\upsilon x}\right) f_1^{''}.$$

$$\Rightarrow \qquad f_1^{'} + \nu U_0 f_1^{'} = \upsilon\left(\frac{\phi}{y}\right) U_0 f_1^{''}$$

2.3 Discussion And Conclusion :

Application of dimensional to solve both ordinary and partial differential equations is illustrated through some well-known examples.

We have seen how writing the odes in such a way that they verify the principle of dimensional homogeneity i.e. introducing dimensional constants, we can obtain in a trivial way change of variable that bring us to obtain simpler ode than the original and therefore their integration is immediate. Furthermore, we have tried to show that these changes of variable are not obtained as if by magic but that they correspond to invariant solutions or to particular solutions and therefore they are generated by the symmetries that admit the ode.

Nevertheless, the dimensional analysis has strong limitations. For example Dimensional analysis is unable (at least at this time we do not know how to do it) to solve the following simple linear ode

$$y^{'} = \left(x^3 + \frac{1}{x} + 3\right)y + \left(3x^2 - \frac{1}{x^2}\right)$$

for this reason one must not put all his confidence in this "tactic". This is one of the greater inconveniences that present the proposed method. But as we have noticed in the introduction, we think that our pedestrian method continues having validity at least when one is studying ode derived from engineering problems or physical problems etc... Where, as it is supposed, such odes must verify the principle of dimensional homogeneity in such a way that for example the ode $y' = x + 1$, lacks of any sense (physical sense, since we cannot add a number to a physical quantity).

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