The Sumudu transform and its application to fractional differential equations

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A B S T R A C T

The Sumudu transform provides powerful operation methods for solving differential and integral equations arising in applied mathematics, mathematical physics, and engineering science. An attempt is made to solve some fractional differential equations using the method of Sumudu transform.
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1. Introduction:

The methods of Integral transforms (Sneddon [8]) have their genesis in nineteenth century work of Joseph Fourier and Oliver Heaviside. The fundamental idea is to represent a function $f(x)$ in terms of a transform $F(p)$ is

$$F(p) = \int_a^b K(p,x) f(x) dx \tag{1.1}$$

where the functions $K(p,x)$ is called kernel.

There are a number of important integral transforms including Fourier, Laplace, Hankel, Laguerre, Hermit and Mellin transforms. They are defined by choosing different kernels $K(p,x)$ and different values for a and b involved in (1.1).

In 1993, G. K. Watugala [9] introduced the sumudu transform to solve differential equations and control engineering problems. Weerakoon [10] has discussed this transform by deriving the Sumudu transform of partial derivatives.

The Sumudu Transform is defined as follows (Watugala [9]):

If $f(t)$ is a function defined on the Real line, then Sumudu Transform of $f(t)$ is defined by

$$F(p) = S[f(t)] = \int_0^\infty \frac{e^{-\left(\frac{t}{p}\right)}}{p} f(t)\,dt \; ; \mathrm{Re}(p) > 0 \tag{1.2}$$

There has been a great deal of interest in fractional differential equations (Miller and Ross [3], Oldham and Spanier [4]). These equations arise in mathematical physics and engineering sciences. There are many definitions of fractional calculus are given by many different mathematicians and scientists (see Podlubny [5]). Here, we formulate the problem in terms of the Caputo fractional derivative (see Caputo [1],[2]), which is defined as:

If α is a positive number and n is the smallest integer greater than α such that $n-1 < \alpha < n$, then the fractional derivative of a function $f(t)$ is defined by (see Podlubny [5]):

$$C^\alpha[f(t)] = \frac{d^\alpha f}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(x)}{(t-x)^{\alpha+n-1}}\,dx \tag{1.3}$$

Further we used the result due to (Katatbeh and Belgacem [6]):

$$S\left\{\frac{d^\alpha f}{dt^\alpha}\right\} = p^{-\alpha} F(p) - \sum_{r=0}^{n-1} p^{r-\alpha} f^{(r)}(0) \tag{1.4}$$

where n is the smallest integer greater than α.

2. Definitions, Notations & Some Results of Special functions

Some special functions used in this thesis are as given below (See Rainville [7]) :

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad \text{where, } \operatorname{Re}(n) > 0 \tag{2.1}$$

Here note that (2.1) can be extended to the rest of the complex plane, excepting zero and negative integer.

Alternative definitions for *Gamma Function* (Rainville [7]) are

$$\Gamma(n) = \lim_{z \to \infty} \frac{z! z^n}{n(n+1)(n+2)...(n+z)} \tag{2.2}$$

$$\Gamma(n) = \frac{e^{-n\gamma}}{n} \prod_{r=1}^\infty \left(1 + \frac{n}{r}\right)^{-1} e^{\frac{n}{r}} \tag{2.3}$$

where γ is known as *Euler's constant* and is defined as following:

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n\right). \tag{2.4}$$

The *Pochhammer symbol* $(\alpha)_n$ (Rainville [7]) is defined by the equations

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1), \, n \geq 1 \tag{2.5}$$

which is Generalization of factorial function.

If α is neither zero nor a negative integer, then we can define $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. (2.6)

The *ordinary binomial expression* (Rainville [7]), defined as

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n \, z^n}{n!} \, .$$ (2.7)

The *Hypergeometric Function* (Rainville [7]) is defined as

$$_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$ (2.8)

where the series on R.H.S. of (2.8), when c is neither zero nor a negative integer, is absolutely convergent within the circle of convergence $|x| < 1$, and divergent outside it; on the circle of convergence the series is absolutely convergent if $\mathrm{Re}(c-a-b) > 0$.

The Generalized Hypergeometric Function (Rainville [7]) is defined as

$$_pF_q\left(\begin{matrix} a_1, a_2, ..., a_p; \\ b_1, b_2, ..., b_q; \end{matrix} x\right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{x^n}{n!} \, .$$ (2.9)

Here note that:

1. If $p \leq q$, the series (2.9) converges absolutely for every finite x.

2. If $p = q+1$, the series (2.9) converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

3. If $p = q+1$, and $|x| = 1$, the series (2.9) converges absolutely when

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$$\text{Re}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0.$$

4. If $p > q+1$, the series (2.9) diverges for $z \neq 0$, and for $z = 0$ its value is one.

3. The solution of fractional Differential Equations:

In this section, we obtain the solution of some fractional differential Equations using Sumudu Transform.

(A) Consider the fractional Differential Equations is of the form

$$\frac{d^{\frac{5}{2}}x}{dt^{\frac{5}{2}}} + \frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} + \frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} = \cosh t \text{ with initial condition } x(0)=0; x'(0)=0; x''(0)=0 \tag{3.1}$$

Solution: Applying the Sumudu Transform (1.2) on both the sides of equation (3.1) and using (1.4), we have

$$\left(p^{-\frac{5}{2}} + p^{-\frac{3}{2}} + p^{-\frac{1}{2}}\right) X(p) - p^{-\frac{5}{2}}x(0) - p^{-\frac{3}{2}}x'(0) - p^{-\frac{1}{2}}x''(0)$$

$$-p^{-\frac{3}{2}}x(0) - p^{-\frac{1}{2}}x'(0) - p^{-\frac{1}{2}}x(0) = \frac{1}{1-p^2} \tag{3.2}$$

Further simplification yields,

$$X(p) = \frac{\dfrac{1}{1-p^2}}{\left(p^{-\frac{5}{2}} + p^{-\frac{3}{2}} + p^{-\frac{1}{2}}\right)} \tag{3.3}$$

Taking inverse Sumudu Transform of (3.3) gives

$$x(t) = -\frac{1}{6\sqrt{\pi}} e^{-\frac{1}{2}t} \int_0^t \frac{e^{\frac{1}{2}u}\left(\cos\left(\frac{1}{2}\sqrt{3}(t-u)\right) - \sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}(t-u)\right)\right)}{\sqrt{u}}\,du$$

$$-\frac{1}{\sqrt{3\pi}} e^{-\frac{1}{2}t} \int_0^t \frac{\sin\left(\frac{1}{2}\sqrt{3}(t-u)\right)e^{\frac{1}{2}u}}{\sqrt{u}}\,du + \frac{1}{6}e^t \mathrm{erf}\left(\sqrt{t}\right)$$

$$-\frac{1}{6\sqrt{\pi}} e^{-\frac{3}{2}t} \int_0^t \frac{e^{\frac{1}{2}u}\left(3\cos\left(\frac{1}{2}\sqrt{3}(t-u)\right) + \sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}(t-u)\right)\right)}{\sqrt{u}}\,du$$

$$+\frac{1}{\sqrt{3\pi}} e^{-\frac{3}{2}t} \int_0^t \frac{\sin\left(\frac{1}{2}\sqrt{3}(t-u)\right)e^{\frac{1}{2}u}}{\sqrt{u}}\,du$$

$$-i\frac{1}{2}e^{-t}\mathrm{erf}\left(i\sqrt{t}\right) \tag{3.4}$$

Equation (3.4) is the solution of (3.1)

Where $\mathrm{erf}(t)$ is the well-known error function (see Rainville [7]).

(B) Consider the fractional Differential Equations is of the form

$$\frac{d^{\frac{3}{2}}y}{dt^{\frac{3}{2}}} + 2\frac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}} = \log_{10} t \text{ with initial condition } y(0)=1; y'(0)=1 \tag{3.5}$$

Solution: Applying the Sumudu Transform (1.2) on both the sides of equation (3.5) and using (1.4), we have

$$\left(p^{-\frac{3}{2}} + 2p^{-\frac{1}{2}}\right)Y(p) - p^{-\frac{3}{2}}y(0) - p^{-\frac{1}{2}}y'(0) - 2p^{-\frac{1}{2}}y(0) = -\gamma + \log_e p \tag{3.6}$$

Further simplification yields,

$$Y(p) = \frac{-\gamma + \log_e p + p^{-\frac{3}{2}} + 3p^{-\frac{1}{2}}}{\left(p^{-\frac{3}{2}} + 2p^{-\frac{1}{2}}\right)} \tag{3.7}$$

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Taking inverse Sumudu Transform of (3.7) gives

$$y(t) = -\sqrt{\frac{t}{\pi}}\gamma - \frac{2e^{-2t}}{\sqrt{\pi}}\int_0^t \sqrt{u}\left(2-\gamma-\ln(4u)\right)e^{2u}du - \delta(t) - \delta'(t) + \frac{e^{-2t}}{4}\left(-i\sqrt{2}\operatorname{erf}\left(i\sqrt{2t}\right)\gamma + 8\right)$$

…………...(3.8)

Where $\delta(t)$ is the well-known Dirac Delta function (see Rainville [7]).

Equation (3.8) is the solution of (3.5)

(C) Consider the fractional Differential Equations is of the form

$$\frac{d^{\frac{7}{2}}z}{dt^{\frac{3}{2}}} + 2\frac{d^{\frac{5}{2}}z}{dt^{\frac{1}{2}}} + \frac{d^{\frac{3}{2}}z}{dt^{\frac{1}{2}}} = e^t \text{ with initial condition } z(0)=1; z'(0)=1; z''(0)=1; z'''(0)=1 \quad (3.9)$$

Solution: Applying the Sumudu Transform (1.2) on both the sides of equation (3.9) and using (1.4), we have

$$\left(p^{-\frac{7}{2}} + 2p^{-\frac{5}{2}} + p^{-\frac{3}{2}}\right)Z(p) - p^{-\frac{7}{2}}z(0) - p^{-\frac{5}{2}}z'(0) - p^{-\frac{3}{2}}z''(0) - p^{-\frac{1}{2}}z'''(0)$$

$$-2p^{-\frac{5}{2}}z(0) - 2p^{-\frac{3}{2}}z'(0) - 2p^{-\frac{1}{2}}z''(0) - p^{-\frac{3}{2}}z(0) - p^{-\frac{1}{2}}z'(0) = \frac{1}{1-p}$$

(3.10)

Further simplification yields,

$$Z(p) = \frac{\dfrac{1}{1-p} + p^{-\frac{7}{2}} + 3p^{-\frac{5}{2}} + 4p^{-\frac{3}{2}} + 4p^{-\frac{1}{2}}}{\left(p^{-\frac{7}{2}} + 2p^{-\frac{5}{2}} + p^{-\frac{3}{2}}\right)}$$

(3.11)

Taking inverse Sumudu Transform of (3.11) gives

$$z(t) = -\sqrt{\frac{t}{\pi}} + \frac{1}{4}\mathrm{erf}\left(\sqrt{t}\right)e^t + \frac{1}{4}\left(4 - i\mathrm{erf}\left(i\sqrt{t}\right) + 8t\right)e^{-t}$$

$$- \frac{4t^{\frac{5}{2}}}{15\sqrt{\pi}} \,_1F_1\left\{2,\frac{7}{2};-t\right\} + \delta'(t) + \delta(t) \tag{3.12}$$

Hence, (3.12) is the required solution of (3.9).

(D) Consider the fractional Differential Equations is of the form

$$2\frac{d^{\frac{7}{2}}z}{dt^{\frac{3}{2}}} + 3\frac{d^{\frac{5}{2}}z}{dt^{\frac{1}{2}}} + \frac{d^{\frac{3}{2}}z}{dt^{\frac{1}{2}}} = \sin t \text{ with initial condition } z(0) = z'(0) = z''(0) = z'''(0) = 0 \tag{3.13}$$

Solution: Applying the Sumudu Transform (1.2) on both the sides of equation (3.13) and using (1.4), we have

$$\left(2p^{-\frac{7}{2}} + 3p^{-\frac{5}{2}} + p^{-\frac{3}{2}}\right)A(p) - 2p^{-\frac{7}{2}}z(0) - 2p^{-\frac{5}{2}}z'(0) - 2p^{-\frac{3}{2}}z''(0) - 2p^{-\frac{1}{2}}z'''(0)$$

$$-3p^{-\frac{5}{2}}z(0) - 3p^{-\frac{3}{2}}z'(0) - 3p^{-\frac{1}{2}}z''(0) - p^{-\frac{3}{2}}z(0) - p^{-\frac{1}{2}}z'(0) = \frac{p}{p^2+1} \tag{3.14}$$

Further simplification yields,

$$A(p) = \frac{\dfrac{p}{p^2+1}}{\left(2p^{-\frac{7}{2}} + 3p^{-\frac{5}{2}} + p^{-\frac{3}{2}}\right)} \tag{3.15}$$

Taking inverse Sumudu Transform of (3.15) gives

$$150\sqrt{\pi}a(t) = 330\sqrt{t} - 8t^{5/2}\,_1F_2\left\{1;\frac{7}{4},\frac{9}{4};-\frac{t^2}{4}\right\} - 60t^{\frac{3}{2}}\,_1F_2\left\{1;\frac{5}{4},\frac{7}{4};-\frac{t^2}{4}\right\}$$

$$-i75\sqrt{\pi}e^{-t}\mathrm{erf}\left(i\sqrt{t}\right) + i240\sqrt{2}e^{-\frac{t}{2}}\sqrt{\pi}\mathrm{erf}\left(\frac{i\sqrt{2t}}{2}\right) \tag{3.16}$$

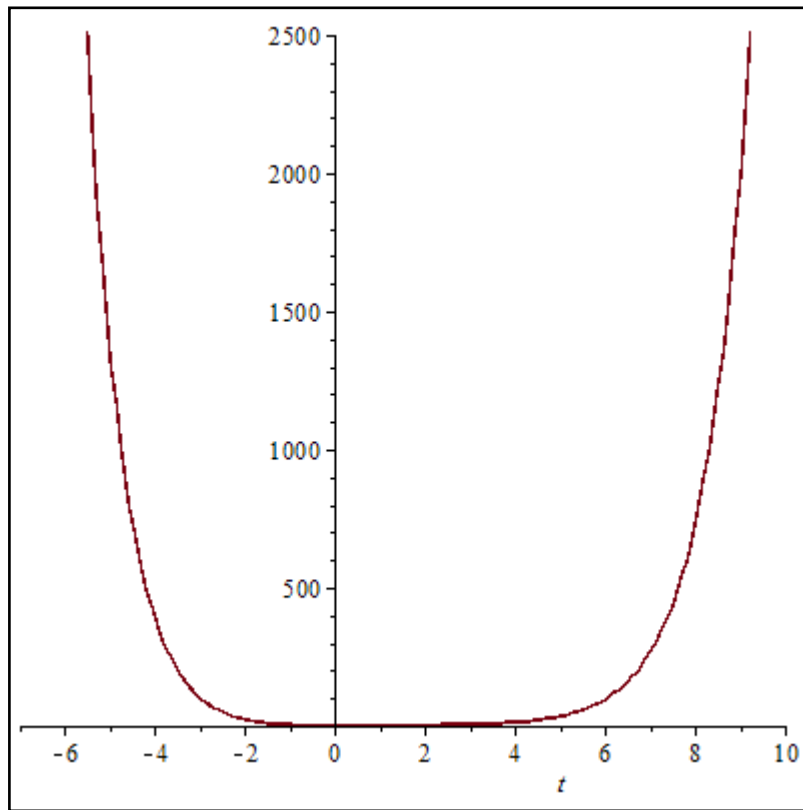Hence, (3.16) is the required solution of (3.13).
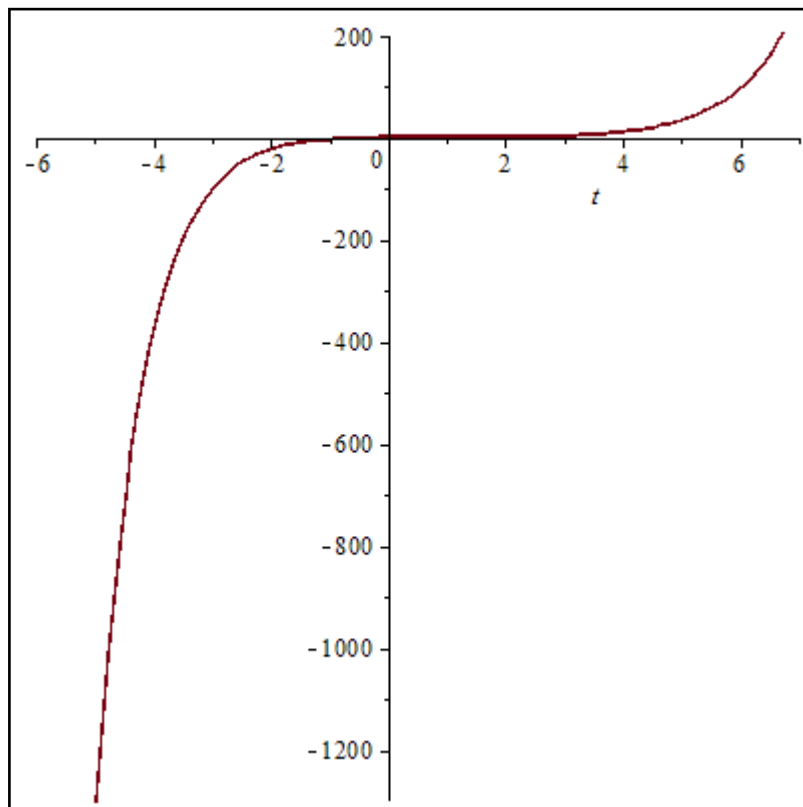
4. Some Graphs:

Figure 4.1: Plot of $|z(t)|$

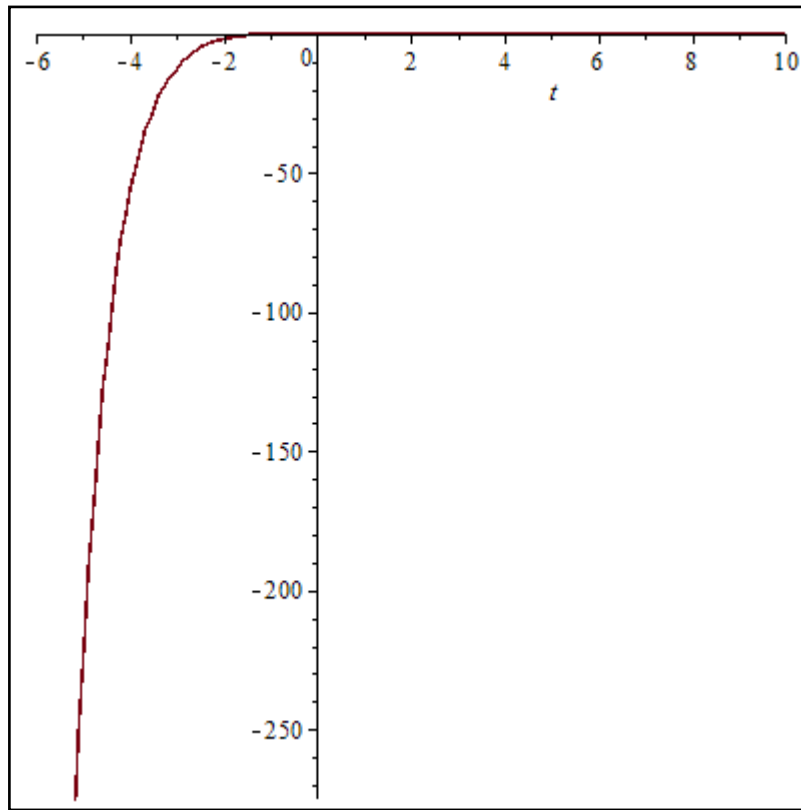Figure 4.2: Plot of $\operatorname{Re}\{z(t)\}$

Figure 4.3: Plot of $\mathrm{Im}\{z(t)\}$
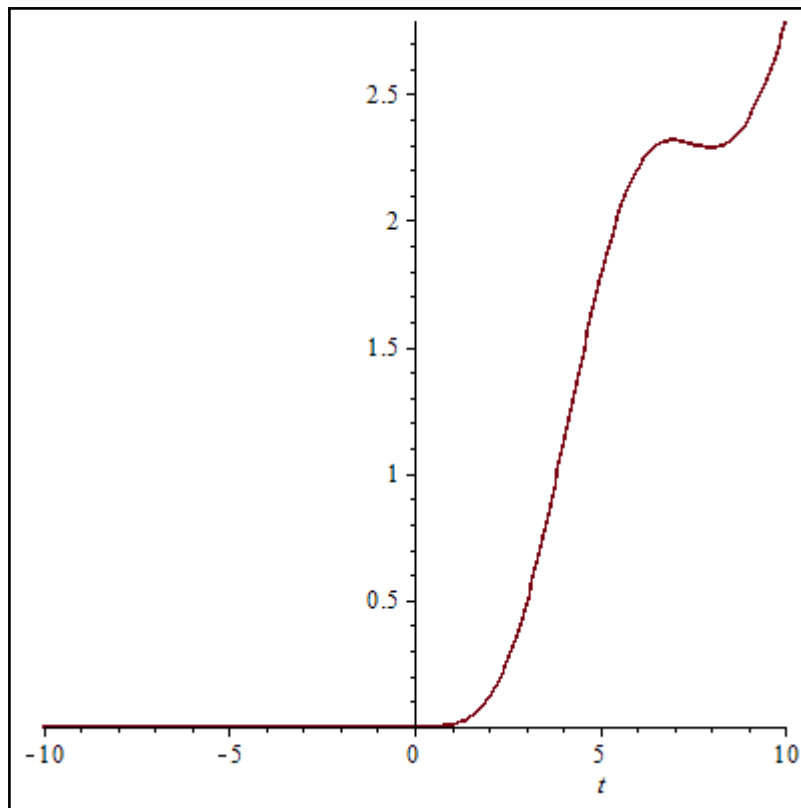
Figure 4.4: Plot of $\mathrm{Re}\{a(t)\}$

Acknowledgement

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