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General Group Transformation Method for the Blasius Boundary Layer Equation with Suction and Injection

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A B S T R A C T

The general group transformations are derived and applied to find similarity solution of well-known Blasius boundary layer equation with suction and injection. In literature, different groups of transformations like Linear group of transformation (or Scaling group of transformation), Spiral group of transformation, Translation group of transformation etc. have been given to find similarity variables (transformations). Here we have proposed one general group of transformation which includes all these three (Scaling, Spiral, Translation) transformations. These techniques can further be extended to transform present similarity equation which is non-linear, third-order ordinary differential equation with three boundary conditions (two at the surface of the plate and one at infinity) into the initial value problem.

The numerical solution of the problem is also discussed. The present technique is hoped to be useful for the boundary value problems of physical sciences and engineering.

Introduction

A fluid is a substance, which deforms continuously subjected to shear forces, however small the shear force may be. The act of continuous deformation is called 'flow'. Fluid dynamics is that branch of mathematics, which deals with the laws of motion of liquids and gases, as well as the interaction of the moving liquids and gases with the solid bodies they flow along. The earlier investigations in the field of fluid dynamics were based on the concept of a perfect (frictionless) fluid. The mathematical theory of perfect fluids is extensively developed and gives satisfactory explanation for some flow phenomenon such as motion of surface waves, the lift and induced drag of an aerofoil. The theory of perfect fluids fails to explain other phenomenon such as skin friction, no slippage etc. In order to understand these we have to study the flow of real fluids, or ordinary fluid dynamics (OFD), which are, in general, viscous and compressible. It was Prandtl (1904) who introduced the concept of boundary layer, so that the Navier-Stokes equations were simplified to a mathematically tractable form, which are then called the Prandtl's boundary layer equations. He made a hypothesis that for fluids with small viscosity (small momentum diffusivity) the flow about a solid body can be divided into two regions:

(i) A very thin layer in the neighborhood of the body where viscous forces are considered, known as the velocity boundary layer or viscous boundary layer and,

(ii) The region outside this layer where the viscous effects may be considered as negligible and the fluid is regarded as in viscid. Thus, when the non-dimensional diffusion parameters such as Reynolds number and Peclet number are large, the phenomenon of boundary layer occurs. The boundary layers are then the velocity and thermal boundary layer and the thickness of the boundary layer is inversely proportional to the square root of the corresponding diffusion number. A detailed coverage of the functions of boundary layer theory has been given in the books by Pai (1956), Meksyn (1961), Rosenhead (1963), Evans (1968), White (1974), Schlichting (1979) , Schlichting and Gersten (1999) and Bansal (1977). A similarity solution is one in which the number of variables can be reduced by one or more by some analytical means, usually by a coordinate transformation or in other words a similarity solution is a solution through which the partial differential equations reduce to ordinary differential equations for which we are able to find accurate numerical solutions.

It will be seen that the similar solutions exist only for a limited choice of the potential flow velocity and the corresponding form of g(x). The concept of similarity solutions and their limitations were introduced by several authors such as Falkner and Skan (1931), Li and Nagamatsu (1953), Kapur and Srivstava (1963), Morgan (1952), Moran and Gaggioli (1967, 68), Gabbert (1967), Banks (1983), Kulkarni et al (1987), Lin and Lin (1987), Williams et al (1987), Pistiner et al (1990), Hussain et al (2000), Sahu et al (2000), Magyari and Keller (2000), Magyari et al (2002), Roy and Takhar (2003), Magyari et al (2003), Mansour and EI-Shaer (2005), Timol et al (1986,1998, 2004) etc and more recently by Timol and his coworkers (2011,2012,2013) have investigated the similarity solutions for MFD and non-MFD boundary layer flow problems.

In the present paper, we have derive general group transformations and applied to find similarity solution of well-known Blasius boundary layer equation with suction and injection. We solve the present Blasius equation by employing combination of two numerical method namely Runge-Kutta-Fehlberg method with shooting technique. The proposed methods have been used extensively by several researchers in dealing with the problems of convective boundary layer flows. The solution procedure is much simpler and effective and the obtained result is of high accuracy.

THE CONCEPT OF AN INFINITESIMAL TRANSFORMATION

Having established the group character of a set of transformations, we now proceed to introduce the concept of an infinitesimal transformation. The discussion here will be limited to a one-parameter group.

A group is said to be continuous if its elements are identified by a set of continuous parameters. Thus, a set of transformation

$$\bar{x} = \phi(x, y, a) \quad ; \quad \bar{y} = \psi(x, y, a) \tag{1}$$

is an example of a one-dimensional continuous group of transformation.

Since ϕ and ψ are continuous functions, the transformation can be written as

$$\bar{x} = \phi(x, y, a_0 + \varepsilon) \quad ; \quad \bar{y} = \psi(x, y, a_0 + \varepsilon) \tag{2}$$

Where a_0 is the value of the parameter corresponding to the identical transformation, that is,

$$\bar{x} = \phi(x, y, a_0) = x \qquad ; \qquad \bar{y} = \psi(x, y, a_0) = y \tag{3}$$

And ε is an infinitesimal quantity which changes x and y by an infinitesimal amount and is defined as an infinitesimal transformation.

Expanding in Taylor series, equation (2) becomes

$$\bar{x} = \phi(x, y, a_0) + \frac{\varepsilon}{1!}\left(\frac{\partial \phi}{\partial a}\right)_{a_0} + \frac{\varepsilon^2}{2!}\left(\frac{\partial^2 \phi}{\partial a^2}\right)_{a_0} + \ldots\ldots\ldots \tag{4a}$$

And

$$\bar{y} = \psi(x, y, a_0) + \frac{\varepsilon}{1!}\left(\frac{\partial \psi}{\partial a}\right)_{a_0} + \frac{\varepsilon^2}{2!}\left(\frac{\partial^2 \psi}{\partial a^2}\right)_{a_0} + \ldots\ldots\ldots \tag{4b}$$

Since ε is an infinitesimal quantity, equation (4) then becomes

$$\bar{x} = x + \varepsilon\xi(x, y) + O(\varepsilon^2) \tag{5a}$$

$$\bar{y} = y + \varepsilon\eta(x, y) + O(\varepsilon^2) \tag{5b}$$

Where

$$\xi(x, y) = \left(\frac{\partial \phi}{\partial a}\right)_{a_0}$$

$$\eta(x, y) = \left(\frac{\partial \psi}{\partial a}\right)_{a_0}$$

And the relation for the identical transformation, equation (3) has been used.

Equation (1) is the "global transformation group" and equation (5) is the "infinitesimal transformation group".

INFINITESIMAL GROUPS OF TRANSFORMATIONS

A given function $f(x, y)$ would be changed to $f(x_1, y_1)$ if it is subjected to the infinitesimal transformation defined by equation (5). Expanding in Taylor series, $f(x_1, y_1)$ becomes

$$f(x_1, y_1) = f(x + \varepsilon\xi, y + \varepsilon\eta) = f(x, y) + \frac{\varepsilon}{1!}Uf + \frac{\varepsilon^2}{2!}U^2 f + \ldots.. \tag{6}$$

Where

$$Uf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}$$

and $U^n f$ represents repeating the operator n times.

Consider the infinitesimal transformation represented by

$$Uf = (c_1x + c_2)\frac{\partial f}{\partial x} + (c_3y + c_4)\frac{\partial f}{\partial y}$$

The transformation functions are given by

$$\xi(x, y) = c_1x + c_2 \quad ; \quad \eta(x, y) = c_3y + c_4$$

To obtain the general group; the following subsystem must be solved

$$\frac{d\bar{x}}{c_1x + c_2} = \frac{d\bar{y}}{c_3y + c_4} = \frac{da}{1} \tag{7}$$

Integrating equation (23), we get:

$$\bar{x} = A^{c_1}x + c_2B \quad ; \quad \bar{y} = A^{c_3}y + c_4B \tag{8}$$

Transformations represented by equation (8) form a "general group of transformation".

THE CONCEPT OF INVARIENCE

A function $f(x, y)$ is said to be invariant under the infinitesimal transformation defined by equation (5), if it is unaltered by the transformation, i.e.,

$$f(\bar{x}, \bar{y}) = f(x, y) \tag{9}$$

Equation (6) shows that (7) will be satisfied if Uf, U^2f, U^3f, etc. is simultaneously equal to zero. However, since $U^2f = U(Uf)$ and $U^3f = U(U^2f)$, it follows that the condition

$$Uf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} = 0 \tag{10}$$

Is both necessary and sufficient requirement for invariance of $f(x, y)$.

Based on the above theorem, the "invariant function" under a given group of transformation represented by Uf can be solved from equation (10).

From elementary theories of partial differential equations, we have

$$\frac{dx}{\xi} = \frac{dy}{\eta} \tag{11}$$

The solution to this equation is given by $\Omega(x, y) =$ constant, which is the required invariant function corresponding to an operator U. Since equation (11) has only one independent solution depending on a single arbitrary constant, a one parameter group of transformations in two variables has one and only one independent invariant.

Application of One-Parameter Group Theory to Boundary-Layer Equations

Consider the following steady, two-dimensional laminar boundary-layer equations for the fluids past a flat plate known as Blasius equation with suction and injection parameter v_w:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{12}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{13}$$

Subject to the boundary conditions

$$y=0: u=0 \ v=v_w$$
$$y=\infty: u=U(x)$$

Group theoretic methods will now be employed to reduce Equations (12) and (13) to ordinary differential equations. To this end, two groups of transformation (namely, the linear and the spiral groups) are considered. The method by Birkhoff and Morgan serves as the basis for the analysis that follows.

The General Group

A general group of transformation is defined by [Patel et al (20 13)]

$$x=A^{\alpha_1}\bar{x}+\alpha_2 b \qquad y=A^{\alpha_3}\bar{y}+\alpha_4 b \qquad u=A^{\alpha_5}\bar{u}+\alpha_6 b$$
$$v=A^{\alpha_7}\bar{v}+\alpha_8 b \qquad U=A^{\alpha_9}\bar{U}+\alpha_{10} b \tag{14}$$

Where $\alpha_1,.....,\alpha_{10}$ are constants and A is the parameter of transformation. The function U will be considered as an independent variable with the defining relation U=U(x). Following the same steps as given in Hansen (1965), we find that the absolute invariants are:

$$\eta=\frac{y}{x^{1/2}};f(\eta)=\frac{u}{x};g(\eta)=\frac{v}{x^{1/2}} \tag{15}$$

And

$$h(\eta)=c_1=\frac{U}{x} \tag{16}$$

Where f and g are functions of η and h (η) = c_1 is a constant since U, the mainstream velocity, is a function of η which would introduce dependence on y.

From Eq. (16), we get the form of mainstream velocity for the establishment of similarity solutions, i.e.

$$U(x)=c_1 x \tag{17}$$

We may now make a check, the transformation of the boundary conditions. It is seen that the boundary conditions are transformed to:

$$\eta = 0 : f = g = 0$$
$$\eta = \infty : f = \frac{U(x)}{x} = c_1.$$

Thus, for constant values of η constant values of f and g result. It is now a simple matter to transform the differential equations. Thus, Equations (12) and (13) are transformed to

$$f' + g' = 0 \tag{18}$$

And $$f'^2 + gf' = \nu f'' \tag{19}$$

We may now make a check, the transformation of the boundary conditions. It is seen that the boundary conditions are transformed to:

$$\eta = 0 : f = 0, g = 1$$
$$\eta = \infty : f = c_1. \tag{20}$$

The above example illustrates the one-parameter group theoretical method. Some further comments are now in order.

In above example, the steps are as follows:

1. Define the group of transformations and substitute into the differential equations.
2. Require that the differential equation be invariant. Relations among the constants in the transformation are to be obtained.
3. Eliminate the parameter of transformation to derive absolute invariants, which will become similarity variables.
4. Apply similarity transformations to transform the Partial differential equations into Ordinary differential equation.
5. Check the boundary conditions to see if they can be transformed into constant values.

From all above five steps, it is seen that here only simple knowledge of differentiation (chain rule) is needed. This makes the group theoretic method an advantage over other methods. As long as in step 2 zeros for all the constants (e.g., $\alpha_1, \ldots\ldots, \alpha_n$ in the linear group) are not obtained, the partial differential equations are transformable into ordinary differential equations.

Step 4 in this particular problem needs not be put before step 5; however, in general, if the unknown function appears in the boundary conditions alone (e.g., the diffusion equation with a boundary condition u(0,t) = U(t) and the form of U(t) is given), it ultimately may save effort to check the boundary conditions first as above.

Equations (12) and (13) can be reduced to one equation by solving Equation (13) for g. However, we are illustrating the method and thus retain f and g in two separate equations.

Numerical Solution

Eliminating g from equations (18) & (19) and without loss of generality setting $c_1 = 1$ and solving resulting third order nonlinear differential equation with the respective boundary conditions numerically by employing the fourth-fifth order Runge-Kutta-Felhberg method with shooting technique using MAPLE software we get numerical results presented in Table 1. We have compared our results with those obtained by Blasius (1908) and Howarth (1938) by the weighted residuals (MWR) method.

CONCLUSION

The group theoretic similarity technique is discussed in detail in this paper. This method is by far the easiest to employ and only a few elementary algebraic steps are required so far, in literature different groups of transformations like Linear group of transformation (or Scaling group of transformation), Spiral group of transformation, Translation group of transformation etc. have been given to find similarity variables (transformations). Here we have proposed one general group of transformation which includes all these three (Scaling, Spiral, Translation) transformations. These techniques can further be extended for system of partial differential equations with three or more independent variables.

It is also found that present results are in good agreement compared to exact solutions by Blasius (1908) for the values $f(\eta)$ and $f'(\eta)$ in comparison with Howarth (1938) as shown in Table 1and Fig. 1-2. The numerical results strongly display the efficiency and accuracy of the proposed method in solving the nonlinear equation.

Table 1

Comparison of the Present results with Published result for $f^{|}(\eta)$

η	Present method	Blasius (1908)	Howarth (1938)
0	0	0	0
0.2	0.0672		

0.4	0.134		
0.5	0.168	0.1659	
0.6	0.201		
0.8	0.269		
1.0	0.330	0.3298	0.32979
1.2	0.395		
1.4	0.455		
1.5	0.485	0.4868	
1.6	0.515		
1.8	0.575		
2.0	0.630	0.6298	0.62977
2.2	0.684		
2.4	0.731		
2.5	0.754	0.7513	
2.6	0.776		
2.8	0.812		
3.0	0.848	0.8460	0.84605
3.2	0.874		
3.4	0.894		
3.5	0.914	0.9130	
3.6	0.919		
3.8	0.935		
4.0	0.955	0.9555	0.95552
5.0	0.989		0.99155
6.0	0.997		0.99898
7.0	1.000		0.99992
8.0	1.000		1.00000
9.0	1.000		1.00000

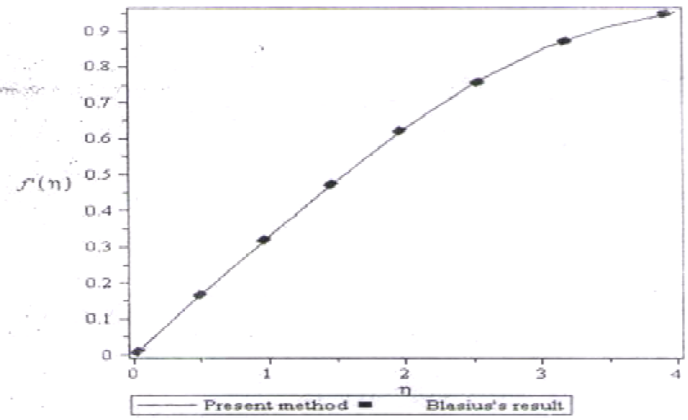

Fig. 1: Comparison of the present results with Blasius [1] for $f'(\eta)$

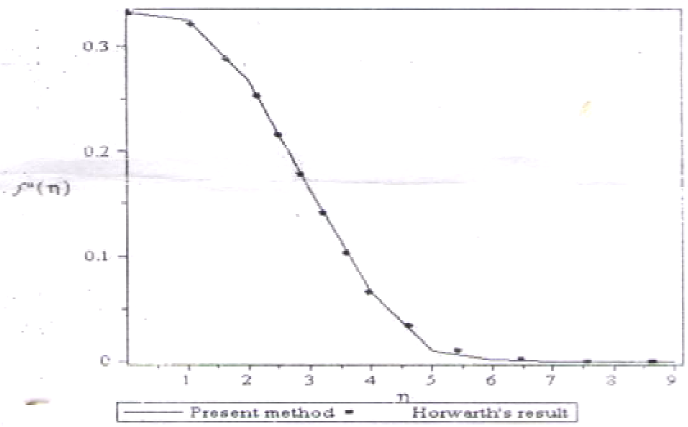

Fig. 2: Comparison of the present results with Horwarth [2] for $f''(\eta)$

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